**Eigenfunctions of L (ODE)**

And now let’s consider angular momentum,

**Angular momentum**

Now let’s consider the angular momentum operator. We will recall from physics 1 that:



and so it follows in quantum mechanics, that if we replace the scalar vectors **r** and **p** with the operator vectors  and , we get the quantum mechanical angular momentum

operator.



and in the position basis, we can write the operator as:



And we see that the components of the angular momentum operator in position space are:



FYI, when you look up these operators in other text books, 99.9% of them will just quote the  expressions. But what they’re showing you is really these operators’ *projections* onto position space, not the actual operator itself. But they still call this the operator. The advantage to their approach is that you don’t waste time with the pesky |**r**><**r**| stuff which goes away anyway when you’re writing down an equation in position space. The disadvantage is that you forget what an operator in the Hilbert space looks really looks like. So I’m writing the |**r**><**r**| to keep everything consistent, and logically correct. But don’t pay too much attention to them. OK, now we’d like to figure out the eigenfunctions of the momentum operator. So we want to solve:



just like we did for the momentum operator. So if we project this operator equation onto the position basis we’ll come to:



similar to what we obtained with the momentum equation. Unfortunately we will find that there is no solution to this equation. This is because there is no function ψ**L**(**r**) that can satisfy all three equations at once. So there is no wavefunction ψ**L** which possesses definite values of Lx, Ly and Lz simultaneously. From a quantum mechanics perspective, this is because the three operators x,y,z don’t commute with each other. Let’s figure out what the commutation relations are:



The middle two commutators are 0 since all of the operators commute with each other. But we have to be more careful with the outer two. Let’s use the identity:



Then our result comes to:



So we get similar results if we work out the other commutation relations…



So generally we get,



where εijk is the Levi-Cevita symbol. Since the operators don’t commute, that means we cannot know all of them simultaneously. If we know one then, we cannot know another with certainty. In fact Heisenberg’s uncertainty principle shows exactly how uncertain our knowledge of the components are. It says that:



**Example**

Show that doesn’t commute with ****. So we must evaluate,



**Example**

Show that it commutes with  though.



**Example**

Show that  and  commute.



So the eigenvalues of these operators can be known simultaneously. So we can try to find eigenvectors of  and . We usually choose i to be equal to 3 (i.e. the z-component). So let’s try to determine the wavefunction which diagonalizes (i.e. is an eigenfunction of) both  and .

**Eigenvectors of** 

First let’s do . We want to solve the equation:



If we project this onto the position basis, we get:



We will want to change variables to spherical coordinates:



The derivatives transform according to:



and the equation transforms to:



This equation is pretty easy to solve. Try an ansatz ψ = Aeiαφ. Plugging this in we have:



So our solution looks like,



Observe that one revolution around the z-axis corresponds to Δφ = 2π. Now in order to ensure that ψ has only *one* value per angle, we need it to repeat itself after every revolution. So we need ψ(φ) = ψ(φ+2π). This requires,



So we see that due to the periodicity condition on the wavefunction, the z-component of the angular momentum (really any component of the angular momentum) is restricted to integer multiples of 2π. So now our wavefunction looks like,



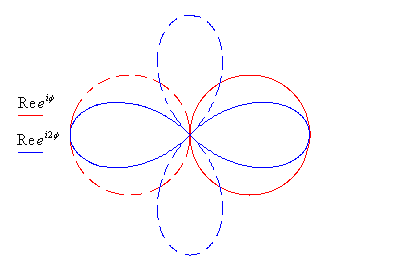
now let’s normalize the wavefunction:



So summarizing,



The m = 1 and m = 2 wavefunctions are plotted below (or rather their real parts). The plots are polar (sort of). Re(ψ) is plotted as the radius of the point on the curve at the angle φ. When Re(ψ) is negative, then dotted lines are used.



Considering the plots for a while, we see that ψ1(φ) oscillates through one wavelength over the φ ε [0, 2π] interval, while ψ2(φ) oscillates through two wavelengths. The number of complete wavelengths is equal to the m quantum number. So the higher the m-number, the smaller the φ wavelength of the wave. Remembering the connection between kinetic energy of a particle and the curvature of its wavefunction we can sort of see that the higher m is, the smaller the wavelength is, the greater the curvature is, the greater speed around the z-axis, the greater the angular momentum around the z-axis – basically.

And since an integer number of wavelengths must fit inside the objects ‘orbit’, it can have only an integer number of ћ angular momentum. Of course since the particle can ‘go’ either clockwise or counter-clockwise around the curve, it can have either positive or negative Lz values.

**Eigenvectors of L2 via Schrodinger equation**

Now we want to determine the eigenfunctions and eigenvalues of the total angular momentum operator. That is want to solve:



We’ll note that since angular momentum was found to be quantized in units of ћ, L2 will be quantized in units of ћ2. So let’s rather write this as:



which allows L2 to now be a dimensionless #, like m. Alright, projecting this equation onto position space.



and switching to spherical coordinates, like we did before, will ultimately give us:



Before proceeding, we’ll remember that Lz and L2 commute. Therefore we can suspect that we can write ψ as the product of the Lz wavefunction and some other function to be determine, which only depends on θ..



Plugging in the ansatz we get,



The next step is to change variables to x = cosθ. Then transforming all the terms…



and so we come to:



So our new equation is:



Notice that the equation is symmetric with respect to m. So the solutions will be the same for positive or negative m. Now we could attempt a series solution at this point. But it would be a pretty ugly series solution. A better approach is to factor out the asymptotic behavior of P first (like we did with the harmonic oscillator equation), and then attempt a series solution. By asymptotic behavior, one means the behavior at the end points of the range of the function. In this case x can range between -1 and 1 so we want to see what P looks like around x = -1 and x = +1. To facilitate this, let’s neglect terms in the ODE which aren’t important near the end points. Clearly the m2/(1-x2) term, which is singular there will be larger than the L2 term so we can neglect the L2 term. One might be tempted to say that the (1-x2)P′′ term is negligible there since it would go to 0, but that supposes that P′′ wouldn’t be singular at the end points and we don’t know that. Same thing goes with P′ in the middle term. So our marginally simplified asymptotic equation is:



Taking a tiny shortcut for time’s sake, looking at the form of the ODE, we may suspect that near the end points P will go something as  where s is some power to be determined, because this is the form it would need, near x = ±1, to cancel out the 1-x2 in the denominotr of the last term, and prevent the derivatives of P from being undefined from the differential equation. Let’s make this assumption and see what s might be. So plugging this ansatz into the ODE we have:



And near this goes to:



So this is the asymptotic behavior, and we choose the + root instead of the negative root because we need P to be finite in the interval. So let’s factor this asymptotic behavior out of the function and change variables to:



Working out the P derivatives…



and filling into our ODE we get:



This is a much simpler ODE. And *now* we’d like to try a series approximation. So let



Then,



Now group powers and even up the summation index,



and



So we have to solve the recursion relation:



There is a neat way to determine the solution to this difference equation for the coefficients. We can form the ratio and then take the ln of both sides. Then we have:



Now sum up both sides from n = 0, 2, 4, …, N-2. Then we have:



Now exponentiating both sides we have:



similarly the odd coefficients would be given by:



So we have formally solved our equation. There are two arbitrary coefficients c0 and c1, which is natural since our ODE was second order and so there should be two linearly independent solutions. Moreover, one solution (setting c0 = something, c1 = 0) has even powers of x. The other (setting c0 = 0, c1 = something) has odd powers of x. So clearly one solution is even and the other odd.

The next question is, though, does the power series converge or not? In fact, we see that it does not necessarily. For, given any fixed L, the product will just keep growing if |m| > 1, as the numerator will eventually be larger than the denominator. So, in order to keep that from happening, we need to keep the numerator smaller than the denominator (for any integer |m|). The only way for this to happen is if the series terminates. If the series is to terminate at cN, then this requires that:



And since N and m are integers, this will require that L2 = ℓ(ℓ+1) where ℓ = N+|m| is a whole number depending on N and |m|. So we have angular momentum (squared) quantization with this condition. We’re not done yet though. Ancillary to this condition is that |m| < ℓ. Otherwise, the numerator can never equal zero. This can be seen in the following. For instance suppose that ℓ = 5. Then the coefficients would be:



Now the series will never terminate if |m| > 5 because the left product in the numerator of the second term, c3, will already be larger than ℓ(ℓ+1). And it will keep growing. So in order for a coefficient to be 0 eventually, we need that the left product start off smaller than ℓ(ℓ+1), which requires |m| < ℓ. One more thing, when ℓ is odd, then N + |m| must be odd. So if m is odd as well, then N must be even, meaning that we have to use the even coefficients. On the other hand, if m is even, then N must be odd and so we have to use the odd coefficients. If ℓ is even though, and m is odd, then N must be odd. And if ℓ is even and m is even, then N must even. So collating these four possibilities, we see that the parity of the polynomial must be (-1)ℓ+|m|. So putting this all together we have:



As stated, c0 and c1 are arbitrary, but there is an arbitrary convention for choosing what they are (like with the Harmonic oscillator Hermite polynomials) that isn’t worth getting into since it only changes the prefactors, which will go away anyway once we normalize the wavefunction. Adjusting c0 or c1 as necessary, you can verify from the formula above that the first few associated Legendre polynomials are:

etc. These associated Legendre polynomials satisfy the orthogonality condition:



and combining the Pℓm(x) with the other factor, eimφ/√2π we get the simultaneous L2, Lz wavefunction:



Normalizing the wavefunctions and calling it Yℓm(θ,φ) we have:



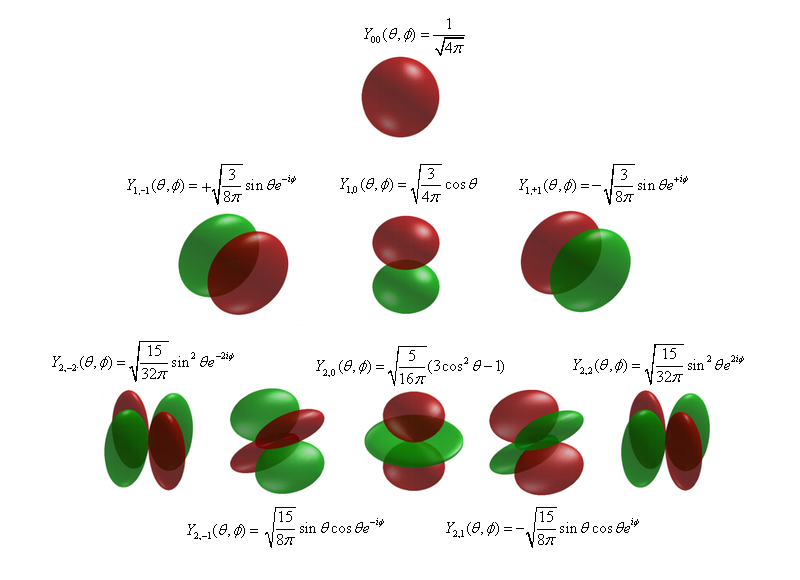
These are the so-called spherical harmonics. These wavefunctions are normalized integrating x (i.e. cosθ) from -1 to 1 and φ from 0 to 2π. So we have:



Remembering what θ and φ represent in spherical coordinates, we’ll recognize that this integration is over the entire surface of a unit sphere. Therefore this is often written,

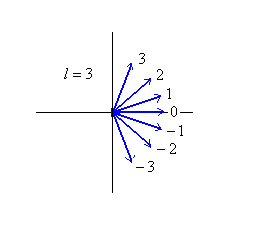


where Ω is the so-called solid angle. This rendering is just basically shorthand for the former. The first few spherical harmonics are depicted below – the images are from Wikipedia. Red points are positive values of Y, and green points are negative values. The magnitude of the Re(Y) at an angle is proportional to the radius of the figure at that angle.



The angular momentum of the wavefunction isn’t entirely evinced by the appearance of the wavefunction. But some gross features can be discerned. First, as ℓ increases, the wavefunctions get curvier. This makes sense because the curviness of ψ is a measure of ψ′s kinetic energy. And the higher magnitude of angular momentum, the higher the kinetic energy so the Y’s ought to get curvier as ℓ increases. Second, the z-component of the angular momentum can be discerned by looking at the wavefunction, Yℓm, from the top down, along the z axis. Let’s keep in mind that Lz is determined by the exp(imφ) factor in Yℓm(θ,φ). Now m would happen to be the # of wavelengths of φ that fit between φ = 0 and 2π. So we can interpret Lz as the number of wavelengths, |m|, that Y oscillates through as we vary φ between 0 and 2π (or rather ћ times this number m). Visually, we can just take a horizontal planar cross-section (essentially a top-down perspective along the z-axis) of the wavefunction Yℓm, and then the number of complete +/- regions, i.e. the # of wavelengths we see, is the |m|-number. Essentially this would be the number of wavelengths or oscillations that Yℓm makes about the z-axis.

Let’s consider another geometric representation of these results. So a particle with angular quantum number ℓ has an angular momentum of ћ√ℓ(ℓ+1). Its z-angular momentum Lz can range between -ℓћ and ℓћ. Its x and y angular momentum would be completely unknown due to Heisenberg’s uncertainty principle – we cannot know Lx and/or Ly if we know Lz because the L operators don’t commute. So the particle would only be able to rotate along the following ‘cones’ (only profile shown) for the case of ℓ = 3. That is, **L** would have to point somewhere along the cone obtained by rotating the arrow about the z-axis. So note that even if m = ℓ, that doesn’t mean that its spin is actually pointing completely in the z-direction (as that would violate the uncertainty principle)



The angles with respect to the x-axis are given by:



since the arrows have a length √ℓ(ℓ+1) and a z-component of m.

**Example**

Suppose we have a particle in a semi-classical state at the origin with position uncertainty of approximately Δr, traveling along the z-axis with momentum expectation ****, and such that:



What is the probability that when we measure its angular momentum we find it to have a magnitude of angular momentum L = ћ, and z-component angular momentum Lz = 0; i.e., what is its overlap (modulus squared) with the state |ψ10>?

Well we first have to normalize the wavefunction so we must compute,



So our normalized wavefunction is equal to, filling in what **p** is…



Alright, now let’s compute its overlap with the state |ψ1,0>. This is:



To evaluate the integral, we should put everything in spherical coordinates.



and now we do the r-integral,



Continuing on…



Finally doing the φ integral we trivially add,

